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Experiments with Equation Solutions by Functional Analysis Algorithms and Formula Manipulation

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The generalizations, due to Kantorovich *et al.* of the well-known numerical algorithms, successive approximations, steepest decent, and Newton's method; onto normed spaces, Hilbert spaces, and Banach spaces, respectively, have been tested on a variety of equations occuring in engineering and physics. Analytical (nonnumerical) solutions to a second-order partial differential equation, a nonlinear first-order ordinary differential equation, van der Pol's equation, a nonlinear damping problem, and a nonlinear two-point boundary value problem were obtained by symbol manipulation as, for example, provided by FORMAC. These algorithms result in relatively simple forms, e.g., polynomials in sines and cosines, depending on the choice of the initial approximation, and yield high accuracy in a few iterations and in seconds-to-minutes of machine time. It is suggested, on the basis of these experiments, that functional analysis algorithms, as developed by Kantorovich, evaluated by automatic formula manipulation can yield analytical solutions of any desired accuracy to a variety of functional equations. In this way, analytical solutions are obtained providing qualitative information while subsequent numerical evaluation avoids much of the art and inaccuracy associated with numerical procedures.

INTRODUCTION

With the recent development of symbol manipulation languages [1, 2] for digital computers, the sheer immensity of formal differentiation, integration, and algebraic manipulation should no longer be an impediment to utilizing many mathematical tools, for example, the exact truncation error,

$$R(x, a, 10) = (1/9!) \int_a^x (x - t)^9 \, (d^{10} \sec t/dt^{10}) \, dt, \qquad x > a,$$

for the Taylor series expansion of sec x about $x = a$ is preferable to the customary bounds used in such calculations. However, this calculation is regarded as impractical due to the enormous, though straightforward, amount of manipulations needed to obtain $R(x, a, 10)$ as an explicit expression in the symbols x and a. The view taken in this paper is that this calculation, and others like it, should be regarded as simple obtainable results by virtue of the symbol manipulation capabilities of large digital computers. In particular, specialized formula manipulation languages such as FORMAC and ALPAK provide this type of operation [1]. An additional viewpoint taken in this paper, is that analytical solutions (non-numerical) of problems are better than numerical ones since they provide qualitative information such as dependence on boundary conditions or on parameters and avoid, as much as possible, the vaguenesses and uncertainties associated with numerical methods. Such must have been the thought of many applied mathematicians of the last century. Probably the best example of this spirit was displayed by Delaunay who over a period of years performed an extensive solution for the motion of the moon by the method of the variation of parameters applied to the perturbative function. Delaunay's highly accurate solution has never been completely used to construct ephemerides, however. Today his solution and solutions to similar problems could be routinely developed and redeveloped as needed for numerical evaluation [5].

The function analysis algorithms, for example, as developed by Kantorovich [3], *et al.* [4, 6] provide a rigorous basis for several general solution methods amenable to formula manipulation and which pertain to a wide range of the equation types occurring in physics and engineering.

Many of the long-known numerical algorithms which have been used efficiently and understood only heuristically have, in comparatively recent times, received a rigorous and useful foundation by examining their extensions from finite Euclidean space to the abstract spaces of functional analysis. Possibly, the most useful example of this extension is Kantorovich's generalization of Newton's method to nonlinear operation between Banach spaces [3].

We will use the generalizations of the methods of successive approximations, steepest descent, and Newton's method to solve, by formula manipulation, some typical partial and nonlinear ordinary differential equations. These solutions were performed by PL/1 programs on an IBM 360/40 or an equivalent computer. We shall, as closely as possible, attempt to use the symbolism in Ref. [3]; in particular, θ shall denote the null element of a metric space and x^* the exact solution to a problem.

Successive Approximations

To examplify the methods of formula manipulation applied to algorithms of functional analysis we start with the methods of successive approximations (SA). Consider the well-known contraction mapping theorem of Banach and Caccioppoli [7].

THEOREM. *Let Ω be a closed subset of a complete metric space X, ρ a metric on X, and P: $\Omega \to \Omega$ a contraction mapping, then there exists a unique solution $x^* \in \Omega$ of $x = P(x)$ and, furthermore, $\{x_n\} \to x^*$, where $x_{n+1} = P(x_n)(n = 0, 1,...)$ for any $x_0 \in \Omega$ with speed of convergence*

$$\rho(x_n, x^*) \leqslant \alpha^n(1 - \alpha)^{-1} \rho(x_0, x_1), (n = 0, 1,...),$$

where α satisfies

$$\rho(P(x), P(y)) \leqslant \alpha\rho(x, y),\ x, y \in \Omega, 0 < \alpha < 1.$$

We apply this theorem to the differential equation $x' - \phi(t, x) = 0$ where $\phi(u, v)$ satisfies the usual existence theorem hypothesis. Let

$$X = C^{(1)}, \Omega = C^{(1)}([0, a])$$

and define

$$P(x(t)) = \int_0^t \phi(x(s), s)\, ds, \qquad x(0) = 0$$

and

$$\rho(x, y) = \max_{t\in[0,a]} | x(t) - y(t)|$$

for this problem. Then we compute

$$\begin{aligned}\rho(P(x), P(y)) &= \max_{t\in[0,a]} \left| \int_0^t [\phi(x(s), s) - \phi(y(s), s)]\, ds \right| \\ &= \max_{t\in[0,a]} \left| \int_0^t \phi_u(\theta(s), s)(x(s) - y(s))\, ds \right| \\ &\leqslant \rho(x, y) \cdot a \max_{\theta,t\in[0,a]} | \phi_u(\theta, t)|\end{aligned}$$

where $\theta(s)$ is the intermediate value of the mean-value theorem. Hence we may assign

$$\alpha = \alpha(a) = a \max_{\theta,t\in[0,a]} | \phi_u(\theta, t)| < 1,$$

where we accomplish $\alpha < 1$ by choosing a sufficiently small. As a concrete example, consider

$$x' - \tanh t \sec x = 0, x(0) = 0,$$

where we seek a solution on the interval $t \in [0, 1]$. We construct an N-degree polynomial solution starting from the zero polynomial $x_0(t) \equiv 0$. By replacing all pertinent functions with their Maclaurin series truncated to the N-th degree, in this case the tan ht and sec x, we insure that each iteration of

$$x_{n+1} = P(x_n) = \int_0^t \tanh s \sec x_n(s)\, ds$$

will result in a polynomial. This iteration has been performed for $N = 10$, 15, and 20, where at each iteration terms of degree higher than N were discarded. The results for $N = 10$ after 3 iterations are shown in Table I. The x-column contains the evaluation of the exact solution,

$$x^*(t) = \sin^{-1}(\ln \cosh t) \text{ for } 0 \leqslant t \leqslant \cosh^{-1} e = 1.656\ldots.$$

Similar results for $N = 15$ and 20 show an increase in accuracy of one and two decimals, respectively.

TABLE I

Solution of x'-tanh t sec $x = 0$, $x(0) = 0$ by Successive Approximations on a 10-th Degree Polynomial

t	x_0	x_1	x	$x_3 - x^*$
.1	0	.004992	.004992	0×10^{-6}
.2	0	.019868	.019869	0
.3	0	.044341	.044355	0
.4	0	.077953	.078032	1
.5	0	.120115	.120405	1
.6	0	.170137	.170967	8
.7	0	.227279	.229273	50
.8	0	.290794	.295014	231
.9	0	.359930	.368086	886
1.0	0	.434329	.448684	2920

We next compare these results with the theoretical speed of convergence. Although the bounds occurring in this and subsequent convergence analysis can be easily shown to exist, they may not, however, be readily evaluated and, frequently, only as extravagant overbounds. This situation can often be remedied by evaluating bounds from a *priori* information obtained from a comparatively crude analog, graphical, or digital solution. For example, for $\phi(u, t) = \tanh t \sec u$ we compute

$$\alpha(a) = a \tanh a \cdot \max_u |\sec u \tan u| = a \tanh a \sec x(a) \tan x(a),$$

where our crude approximate solution (possibly the first iteration) suggests the monotonically increasing nature of $x(a)$, which is evident in this simple example, and where we have evaluated the uniform bound by now assuming Ω to be composed of functions $x(t)$ in $C^{(1)}([0, 1])$ and, in addition, contained in a narrow corridor about the crude approximation. It appears that only with such heuristics is it possible to use theoretical convergence information as an aid to obtaining explicit solutions. A crude solution yields $x(.5) \simeq .12$ and $x(1) \simeq .5$; hence $\alpha(.5) \simeq .028$ and $\alpha(1) \simeq .26$, from which, and from Table 1 we obtain, for $N = 10$,

$$\rho(x_n(.5), x^*(.5)) \leqslant .12(10^{-2.6n})$$
$$\rho(x_n(1), x^*(1)) \leqslant .58(10^{-.6n})$$

which is in fair agreement with Table 1. We note that the polynomials contained in this way converge with corresponding high accuracy to the composite Maclaurin series of the exact solution

$$\begin{aligned} x^*(t) &= \sin^{-1}(\ln \cosh t) \\ &= \sum_{k=0}^{\infty} \left[\frac{1 \cdot 3 \cdots (2k-1)}{2 \cdot 4 \cdots (2k)} \cdot \frac{1}{2k+1} \left(\sum_{j=1}^{\infty} \left[\frac{(-1)^{j+1}}{2j} \sum_{i=1}^{\infty} \frac{t^{2i}}{(2i)!} \right] \right)^{j} \right]^{2k+1} \\ &= (1/2!)\, t^2 + [(1/6!) - (1/2)(1/2!)^2]\, t^4 \\ &\quad + [(1/8!) + (2/3)/(1/2!)^3 - (1/2)(1/2!\, 6!)] + \cdots, \end{aligned}$$

where $-\cosh^{-1} 2 < t \leqslant \cosh^{-1} 2 \simeq 1.313$. In this case, the Maclaurin form for the solution proved efficient; however, other forms of a piecewise construction

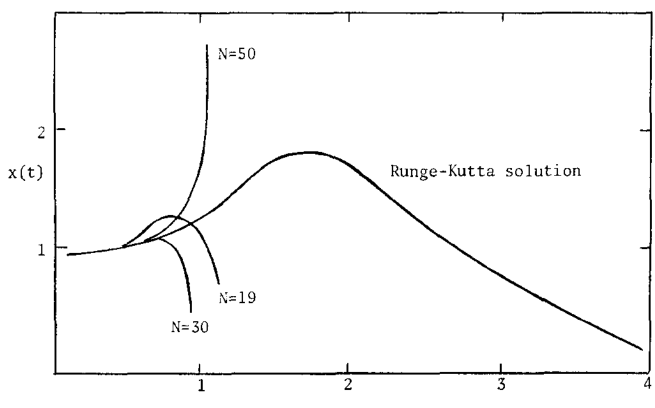

FIG. 1. Solution of $x' - \sin((x + 1)t)$, $x(0) = 0$ by successive approximations on an N-th degree polynomial.

such as spline functions [8] are more generally reliable. The difficulties in using a single polynomial as the solution form, as opposed to polynomial splines, is demonstrated in Fig. 1, where the previous procedure was applied to

$$x' - \sin((x + 1)\,t) = 0,\ x(0) = 0 \text{ for } N = 20, 30, 50.$$

In the range $t \leqslant .5$, accuracy better than 10^{-7} is observed in a few iterations.

Fixed-point theory and the theory of contraction mappings provide an analytical basis for the automatic formula manipulation solution of many generic types of equations; for example, see Ref. [3] and the bibliography there.

Steepest Descent

The method of steepest descent (SD) possesses a more restrictive setting than does SA, in that its rigorous development usually requires that the mappings, associated with equations whose solutions are desired, are defined on Hilbert spaces rather than complete metric spaces. However, the SD method, whether it applies rigorously or not, seems to provide more rapid convergence than SA and apply to a wider variety of equations of engineering and physics.

We describe the SD method: Let ϕ be a real-valued function defined on a normed space X and let it be desired to find $x^* \in X$ such that $\phi(x) \equiv \phi(x^*)(x \in X)$ by constructing a minimizing sequence $\{x_n\}$ with

$$\lim_{n\to\infty} \phi(x_n) = \inf_{x\in X} \phi(x), \qquad \lim_{n\to\infty} x_n = x^*,$$

provided ϕ is continuous with respect to $x \in X$. We define

$$\bar{\phi}(a, x_0, z) = \phi(x_0 + az),$$

where $\{x \mid x = x_0 + ax, a \geqslant 0, z \neq \theta\} \subset X$ and construct $\bar{z}$, the negative gradient, by minimizing

$$\| z \|^{-1} (\partial\bar{\phi}/\partial a)|_{a=0} = \| z \|^{-1} \lim_{a\to 0^+} (\phi(x_0 + az) - \phi(x_0))$$

over all possible directions z. The optimal step size along direction $\bar{z}$ is then obtained as the smallest positive root λ_1 of $\partial\bar{\phi}/\partial a = 0$. This yields the second element of the sequence as $x_1 = x_0 + \lambda_1 z$. This construction is summarized in the SD algorithm shown in Figure 2.

The SD algorithm possesses wide heuristic application since, it seems, a minimum principle exists or can be contrived for nearly all problems of engineering and physics. The variational calculus suggests many of the minimizations related to

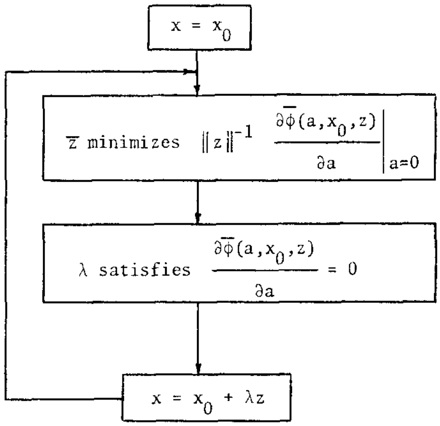

FIG. 2. Steepest descent algorithm.

ordinary and partial differential equations; for example, the problems in Ref. [9]. In many cases, the equation and the minimization problem have been shown to be equivalent in that their solution sets are the same (10–12). An example of the equivalency in a general setting is given by the following theorem [10]:

THEOREM. *Let H be a Hilbert space and $U\colon H \to H$ be a self-adjoint linear operator with lower bound* $m = \inf_{x \neq \theta} (Ux, x)/(x, x) > 0$. *Let $x, y \in H$ and define*

(1) $Ux = y$,
(2) $F(x) = (Ux, x) - (x, y) - (y, x)$,

then the solution $x = x^*$, if it exists, of (1) *minimizes $F(x)$ and, conversely, the*

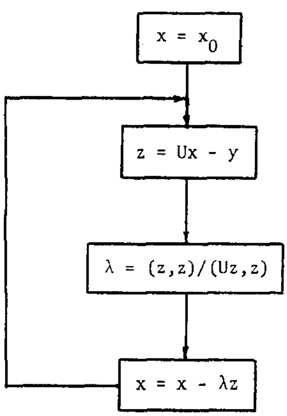

FIG. 3. Steepest descent algorithm for $F(x)$.

element $\tilde{x}$, *if it exists, which minimizes* (2) *satisfies* (1), *i.e.*, $x^* = \tilde{x}$; *furthermore,* grad $F = Ux^* - y = \theta$ *if and only if* x^* *minimizes* $F(x)$.

The SD algorithm applied to $F(x)$ is shown in Fig. 3.

Kantorovich [3] obtains the following convergence theorem for this algorithm:

THEOREM. *$U: H \to H$ is a self-adjoint linear operator on Hilbert space H with bounds*

$$M = \sup_{x \neq \theta} \frac{(Ux, x)}{(x, x)} > m = \inf_{x \neq \theta} \frac{(Ux, x)}{(x, x)} > 0;$$

then the SD algorithm minimizes $F(x)$(i.e., solves $Ux = y$) for fixed $y \in H$ and converges to a unique element $x^* \in H$ with speed of convergence

$$\| x_n - x^* \| \leqslant \frac{\| z_1 \|}{m} \left(\frac{M - m}{M + m} \right)^n, \qquad n = 0, 1, \ldots .$$

As a concrete example of the SD method and one utilizing these theorems, we follow the developments of Kantorovich and Krylov [3, 12] in their treatment of the self-adjoint equation

$$L(x) = -\frac{\partial}{\partial s}\left(a \frac{\partial x}{\partial s}\right) - \frac{\partial}{\partial t}\left(b \frac{\partial x}{\partial t}\right) + cx = d, \qquad x \mid_{\partial D} = 0,$$

where it is assumed that $a(s, t)$ and $b(s, t) \in C^{(1)}(D)$, $c(s, t)$ and $d(s, t) \in C^{(0)}(D)$, and $a(D) > 0$, $b(D) > 0$, $c(D) \geqslant 0$; and we adopt the inner product

$$(x, y) = \int_D \int (y_s x_s + y_t x_t)\, ds\, dt$$

associated with the Sobolev norm $\| \cdot \|_w$ on the space $\dot{W}_2^{(1)}(D)$ of differentially continuous functions over D satisfying the boundary condition and such that $\| x \|_w = (x, x)^{1/2}$ is uniformly bounded over D. The problem is transformed into the form of $F(x)$ by defining

$$Ux \equiv -\Delta^{-1} Lx = -\Delta^{-1} d \equiv y$$

and computing

$$F(x) = (Ux, x) - (x, y) - (y, x) = \cdots = \int_D \int [a x_s^2 + b x_t^2 + c x^2 - 2yx]\, ds\, dt,$$

where Δ^{-1} is the inverse (necessarily linear in this context [3, 13]) of the Laplacian operator, $\Delta x(s, t) = x_{ss} + x_{tt}$. The resulting U has the required properties and

SD algorithm for the solution of $Lx = d$ follows immediately and is shown in Fig. 4.

If the functions a, b, c and d are replaced by their polynomial or piecewise polynomial approximations with the aforementioned properties, and if x_0 is also polynomial, then the manipulations in Fig. 4 result in polynomials. In particular, the solution of $-\Delta z = Lx - d, z|_{\partial D} = 0$ is a polynomial of the form

$$z(s, t) = (s^2 + t^2 - 1)\, p(s, t),$$

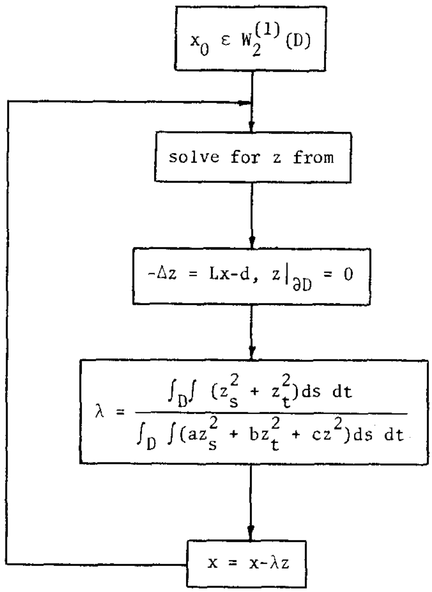

FIG. 4. Steepest descent algorithm for $Lx = d$.

where D has been taken as the unit circle $D = \{(s, t)|\, s^2 + t^2 = 1\}$, and where $p(s, t)$ is the general polynomial in symbols s and t of degree two greater than the degree of Lx-d. The coefficients of $p(s, t)$ can be explicitly obtained symbolically from the method of undetermined coefficients. Similarly, the integrals in the expression for λ can be explicitly manipulated; then evaluated exactly.

Formula manipulation based on polynomial operations has been performed for the case where $a = 1$, $b = 2(3 + s^2 + 4t^2)$, $d = 2e(3 + s^2 + 4t^2)$ and

$$\{(s, t)|\, s^2 + t^2 \leqslant 1\} = D$$

starting with $x_0 = 2 - 2s^2 - 2t^2$. The results of these calculations are shown in Table 2 for polynomial solutions of degree $N = 10, 15$ and 20 and where iterations beyond the $n = 3, 4$ and 5, respectively, produced no further improvement. The exact solution is given by

$$x^*(s, t) = e - e^{s^2+t^2}.$$

The execution time to construct Table II was 8 sec on a CDC 3300.

TABLE II
The accuracy of polynomial solutions of $Lx = d$

$s = 0$	$N = 10$	$N = 15$	$N = 20$
t	$x_3 - x^*$	$x_4 - x^*$	$x_5 - x^*$
0	0×10^{-6}	0×10^{-6}	0×10^{-6}
.1	0	0	0
.2	0	0	0
.3	0	0	0
.4	1	0	0
.5	1	0	0
.6	8	0	0
.7	50	4	0
.8	231	29	−1
.9	886	178	−14
1.0	2920	895	−176

Theoretical lower and upper bound $\bar{m}$ and $\bar{M}$ on m and M for $Lx = d$ are found in Ref. [3];

$$0 < \bar{m} \leqslant m < M \leqslant \bar{M}.$$

However, the $\bar{m}$ and $\bar{M}$ of [3] are difficult to compute and seem to be unreasonably poor approximations of m and M. If once again we use a crude approximate solution, such as the first iteration or an independent approximation, to hopefully narrow down the solution space to a subset in $\dot{W}_2^{(1)}(D)$, then m and M can be estimated. From [3] we have

$$\bar{m} = \min[\min_D a, \min_D b],$$

$$\bar{M} = 2 \max[\max_D a, \max b, A \max_D c],$$

where

$$A \geqslant \| x \|_W^{-2} \int_D \int | x |^2 \, ds \, dt, \qquad x \in D.$$

If A is estimated by selling $x = x_0$, then the speed of convergence is set by

$$\left(\frac{\overline{M} - \overline{m}}{\overline{M} + \overline{m}}\right)^n \simeq (.96)^n.$$

However, if $x_1 + | x_1 - x_0 |$ and $x_1 - | x_1 - x_0 |$ are thought of as bounding surfaces for x^*, then we may compute

$$\frac{(Ux, x)}{(x, x)} = \frac{\int_D \int (ax_s^2 + bx_t^2 + cx^2)\, ds\, dt}{\int_D \int (x_s^2 + x_t^2)\, ds\, dt},$$

directly for these two extremes and assign $\overline{M}$ to the larger such number and $\overline{m}$ to the smaller. In our case the speed of convergence becomes

$$\left(\frac{\overline{M} - \overline{m}}{\overline{M} + \overline{m}}\right)^n \simeq (.25)^n.$$

Since the computing time per iteration is small, less than 1 sec, accurate solutions can be obtained even where very large n are required. Experiments with the method of SD on algebraic linear systems with difficult to invert matrices have resulted in accurate solutions after from 50 to 100 iterations [14] of Kantorovich's p-step variant of the SD algorithm [3, 15]. The p-step variant would, surely, correspondingly increase the speed of convergence for the general equation satisfying the above theorems, in fact, Kantorovich has shown the convergence factor becomes $[(M - m)/(M + m)]^{np}$ where p is a positive integer. The above-mentioned experiment with $p = 2, 3, \ldots, 10$ clearly showed the efficiency of large p.

The Frechet Derivative

Many types of generalized derivatives have been invented to study nonlinear operators [16, 17]. The Frechet derivative permits the generalization of Newton's method:

Definition. Let X and Y be Banach spaces, open $\Omega \subset X$, $\Omega' \subset Y$, $x_0 \in \Omega$, $P : \Omega \to \Omega'$, define

$$P'(x_0)(x) = \lim_{t \to 0} \frac{1}{t} [P(x_0 + tx) - P(x_0)],$$

if $P'(x_0)$ is a linear operator in $[X \to Y]$ and if the limit converges uniformly for $\{x \mid \| x \| = 1, x \in X\}$, then the operator $P'(x_0)$ is the Frechet derivative of $P(x)$ and $P'(x_0)(x)$ is the Frechet differential.

Similarly the n-th Frechet derivative is an n-linear operator which can be constructed by n successive differentiations, for example, the second derivative can be computed from

$$P''(x_0)(x, \bar{x}) = \lim_{t\to 0} \frac{1}{t} [P'(x_0 + t\bar{x})(x) - P'(x_0)(x)],$$

or

$$P''(x_0)(x, \bar{x}) = \lim_{t'\to 0} \lim_{t\to 0} (tt')^{-1} [P(x_0 + t'x + t\bar{x}) - P(x_0 + t'x) - P(x_0 + t'\bar{x}) + P(x_0)].$$

Note that if $P(x)$ is linear then $P'(x_0)(x) = P(x)$ and $P''(x_0)(x, \bar{x}) = \theta$. As an example, consider

$$P(x) = x'' + x + \phi(t, x, x')$$

which by Taylor's series with obvious assumptions on $\phi(t, u, v)$ yields

$$\begin{aligned} P'(x_0)(x) &= x'' + x + P'(x_0)(\phi(t, x, x')) \\ &= x'' + x + \lim_{t\to 0} \frac{1}{t} [\phi(t, x_0 + tx, x_0' + tx') - \phi(t, x_0, x_0')] \\ &= x'' + x + \lim \frac{1}{t} [\phi(t, x_0, x_0') + \phi_u(t, x_0, x_0')\, tx \\ &\quad - \phi_v(t, x_0, x_0')\, tx' - \phi(t, x_0, x_0') \\ &= x'' + x + \phi_u(t, x_0, x_0')\, x + \phi_v(t, x_0, x_0')\, x'. \end{aligned}$$

Similarly,

$$P''(x_0)(x, \bar{x}) = \phi_{uu}(t, x_0, x_0')\, x\bar{x} + \phi_{uv}(t, x_0, x_0')(x\bar{x}' + x\bar{x}) + \phi_{vv}(t, x_0, x_0')\, x'\bar{x}'.$$

We see that the operators $P'(x_0)$ and $P''(x_0)$ have the required properties. The inverse Frechet derivative is the solution of $P'(x_0)(x) = y$ and will be written $[P'(x_0)]^{-1} y$. It is clear, that except for simple forms of $\phi(t, u, v)$ and/or for judicious choices of x_0, the explicit construction of this inverse is a forbidding task; for instance, in the above example one must formally solve a second-order differential equation with variable coefficients. In the following sections on Newton's method choices of x_0 and simple transformations of $P(x)$ will be used enabling $[P'(x_0)]^{-1}$ to be exhibited and, furthermore, to be linear in $[Y \to X]$.

Newton's Method

Newton's method, or the method of tangents, has been long used for computing real or complex zeros of real or complex equations. The extension of this concept to nonlinear operators on Banach spaces is mainly due to Kantorovich [3, 15]. The form of the generalization is immediate; if $P: X \to Y$ is a nonlinear operator between Banach spaces, then one is led to considering the following algorithms for computing x^* such that $P(x^*) = \theta$,

$$x_{n+1} = x_n - [P'(x_n)]^{-1} P(x_n) \quad \text{or} \quad P'(x_n)(x_{n+1} - x_n) = -P(x_n),$$
$$x_{n+1} = x_n - [P'(x_0)]^{-1} P(x_n) \quad \text{or} \quad P'(x_0)(x_{n+1} - x_n) = -P(x_n),$$
$$x_{n+1} = x_n - \Gamma P(x_n).$$

The first algorithm is known as the original Newton's method (ONM), the second the modified Newton's method (NMN), and the last we shall call the generalized Newton's method (GNM). The GNM represents an additional simplification of the ONM in that Γ is a constant operator approximating $[P'(x_n)]^{-1}$ but not requiring the construction of $P'(x_0)$ or $[P'(x_0)]^{-1}$ as does the MNM. A summary of Kantorovich's theory, as found in Ref. [3], is given in Table III. These theorems provide information concerning the uniqueness, existence, speed of convergence, and the region of accessibility from the initial approximation x_0. Additional results on the GNM with less restrictive hypotheses can be found elsewhere; for example, see Antosiewicz [18]. There are numerous examples of Newton's method used in a totally numerical manner to solve functional equations [6, 19, 20, 22, 23].

Theorems 7 and 8 and a special case of Theorem 13 will prove to be of special interest in applying Newton's method to ordinary differential equations. We note the special case of Theorem 13 formed by decomposing $P(x)$ in Theorem 8 into

$$P(x) = \pi(x) + R(x), \pi(x_0) = \theta,$$

where the solution of $\pi(x) = \theta$ is 'close' to that of $P(x) = \theta$ and where $[\pi'(x_0)]^{-1}$ is linear. If in Theorem 7 we set $\Gamma = [\pi'(x_0)]^{-1}$, then the hypothesis conditions and conclusions become, respectively,

H(3)′ $\quad \| \Gamma P(x_0)\| = \|[\pi'(x_0)]^{-1} R(x_0)\| \leqslant \eta$

H(4)′ $\quad \| \Gamma P'(x_0) - I \| = \|[\pi'(x_0)]^{-1} R'(x_0)\| \leqslant \alpha < 1$

H(5)′ $\quad \|[\pi'(x_0)]^{-1} \pi''(x)\| \leqslant K \quad (x \in \Omega), \qquad \|[\pi'(x_0)]^{-1} R''(x)\| \leqslant L \quad (x \in \Omega)$

H(6)′ $\quad h = (K + L)(1 - \alpha)^{-2} \qquad \eta \leqslant 1/2$

H(7)′ $\quad r \geqslant r_0 = (1 - \sqrt{1 - 2h})(1 - \alpha)^{-1} \eta/h$

C(2)′ $\quad \| x_n - x^* \| \leqslant 2^{-n}(2h)^{2n} (1 - \alpha)^{-1} \eta/h,$ ONM

C(4)′ $\quad \| x_n - x^* \| \leqslant [1 - (1 - \alpha) \sqrt{1 - 2h}]^{n+1} (1 - \alpha)^{-2} \eta/h,$ MNM

TABLE III

Theorems on Newton's Method

Legend: X and Y are B-spaces
$\Omega_R \equiv \{x \mid \| x - x_0 \| < R\} \subset X$
$\Omega_0 \equiv \bar{\Omega}_r \equiv \{x \mid \| x - x_0 \| < r\} \subset X, (r < R)$

$P\colon \Omega_R \to Y$
$S\colon \Omega_R \to Y$
$\phi\colon [t_0, t'] \to R' \quad (t' - t_0 \equiv r < R), (\phi(t) = t + C_0\psi(t))$

$x_0 \in \Omega_0$: initial approximate solution to $x = S(x)$ of $P(x) = \theta$
$x_n \in X$: iterates
$x^* \in \Omega_0$: solution to $x = S(x)$ or $P(x) = \theta$
t^*: least root of $\theta(t)$ of $\psi(t)$ in $[t_0, t']$, t_0 corresponds to x_0.

Successive approximations (SA) for $x = S(x)$: $x_{n+1} = S(x_n)$
Modified Newton's method (MNM) for $P(x) = \theta$: $x_{n+1} = x_n - [P'(x_0)]^{-1}P(x_n)$
Original Newton's method (ONM) for $P(x) = \theta$: $x_{n+1} = x_n - [P'(x_n)]^{-1}P(x_n)$
Generalized Newton's meth(GNM) for $P(x) = \theta$: $x_{n+1} = x_n - \Gamma P(x_n)$

Theorem	Hypothesis	Conclusion
Def: majorant function	(1) $\| S(x_0) - x_0 \| \leqslant \phi(t_0) - t_0$ (2) $\| x - x_0 \| \leqslant t - t_0 \Rightarrow \| S'(x) \| \leqslant \phi'(t)$	(1) ϕ majorizes S in $[t_0, t']$.
Th(1): SA	(1) S has cont. der. in Ω_0 (2) ϕ is diff. in $[t_0, t']$ (3) ϕ has a root in $[t_0, t']$ (4) ϕ majorizes S (5) $x_0 \in \Omega_0$.	(1) There exists $x^* \in \Omega_0$ (2) SA conv. to x^* for every $x_0 \in \Omega_0$ with $x_n \in \Omega_0$ (3) $\| x^* - x_0 \| \leqslant t^* - t_0$
Th(2): SA Uniqueness	(1–5) Same as (1–5) in Th(1) (6) ϕ has unique sol. in $[t_0, t']$ (7) $\phi(t') \leqslant t'$	(1–3) Same as (1–3) in Th(1) (4) x^* is unique in Ω_0
Th(3): MNM	(1) P has cont. second der. on $\bar{\Omega}_R$. (2) $\psi \in C^{(2)}[t_0, t']$ such that: (3) there exists $\Gamma_0 \equiv [P'(x_0)]^{-1} \in [Y \to X]$ (4) $C_0 \equiv -1/\psi'(t_0) > 0$ (5) $\| \Gamma_0 P(x_0) \| \leqslant C_0\psi(t_0)$ (6) $\| x - x_0 \| \leqslant t - t_0 \Rightarrow \| \Gamma_0 P''(x) \| \leqslant C_0\psi''(t)$ (7) ψ has a root in $[t_0, t']$	(1) there exists $x^* \in \Omega_0$ (2) MNM conv. to x^* for every $x_0 \in \Omega_0$ with $x_n \in \Omega_0$ (3) $\| x^* - x_0 \| \leqslant t^* - t_0$
Th(4): MNM Uniqueness	(1–7) Same as (1–7) in Th(3) (8) $\psi(t') \leqslant 0$ (9) $\psi(t)$ has unique sol. in $[t_0, t']$	(1–3) Same as (1–3) in Th(3) (4) x^* is unique in Ω_0

Table continued

TABLE III (*continued*)

Theorem	Hypothesis	Conclusion
Th(5): ONM	(1–7) Same as (1–7) in Th(3)	(1–3) Same as (1–3) in Th(3) for ONM.
Th(6): ONM Uniqueness	(1–9) Same as (1–9) in Th(4)	(1–3) Same as (1–3) in Th(5) (4) x^* is unique in Ω_0
Th(7): MNM and ONM for quadratic majorant	(1) P has cont. second der. in Ω_0 (2) there exists $\Gamma_0 \equiv [P'(x_0)]^{-1} \in [Y \to X]$ (3) $\\| \Gamma_0 P(x_0) \\| \leqslant \eta$ (4) $x \in \Omega_0 \Rightarrow \\| \Gamma_0 P''(x_0) \\| \leqslant K$ (5) $h \equiv K\eta \leqslant 1/2$ (6) $r \geqslant r_0 \equiv (1 - \sqrt{1-2h})\eta/h$	(1) there exists $x^* \in \Omega_0$ (2) MNM or ONM conv. to x^* for every $x_0 \in \Omega_0$ with $x_n \in \Omega_0$ (3) $\\| x^* - x_0 \\| \leqslant r_0$ (4) for MNM, ONM, resp.; $\\| x^* - x_n \\| \leqslant 2^{-n}(2h)^{2n}\eta/h$ $\\| x^* - x_n \\|$ $\leqslant (1 - \sqrt{1-2h})^{n-1}\eta/h$ if $h < 1/2$
Uniqueness	(7) If, in addition, $h < 1/2$ and $r < r_1 \equiv (1 + \sqrt{1-2h})\eta/h$ or $h = 1/2$ and $r \leqslant r_1$.	(5) x^* is unique in Ω_0
Th(8): Alternate form of Th(7)	(1) $\\| \Gamma_0 \\| \leqslant B'$ (2) $\\| P(x_0) \\| \leqslant \eta'$ (3) $x \in \Omega_0 \Rightarrow \\| P''(x) \\| \leqslant K'$ $h = K'B'^2\eta'$ $\eta = B'\eta'$ $K = B'K'$ $r_0, r_1 = (1 \mp \sqrt{1-2h})B'\eta'/h$ (4–7) Same as (1, 4–6) in Th(7)	(1–4) Same as (1–4) in Th(7)
Uniqueness	(8) Same as (7) in Th(7)	(5) Same as (5) in Th(7)
Th(9): GNM	(1) P has cont. second der. in Ω_0 (2) $\Gamma \in [Y \to X]$ such that: (3) $\\| \Gamma P(x_0) \\| \leqslant \bar\eta$ (4) $\\| \Gamma P'(x_0) - I \\| \leqslant \delta$ (5) $x \in \Omega_0 \Rightarrow \\| \Gamma P''(x) \\| \leqslant \bar K$ (6) $\bar h \equiv \bar K \bar\eta (1-\delta)^{-2} \leqslant 1/2 \quad \delta < 1$ (7) $r \geqslant \bar r_0 \equiv (1 - \sqrt{1-2h})(\bar\eta/\bar h)(1-\delta)^{-1}$	(1) there exists $x^* \in \Omega_0$ (2) for ONM $\\| x^* - x_n \\| \leqslant 2^{-n}(2\bar h)^{2^n}(\bar\eta/\bar h)(1-\delta)^{-1}$ (3) GNM conv. to x^* for every $x_0 \in \Omega_0$ with $x_n \in \Omega_0$ (4) for GMN for $h < 1/2$
Uniqueness	(8) If in addition $h < 1/2$ and $r < \bar r_1 \equiv (1 + \sqrt{1-2h})(\bar\eta . \bar h)(1-\delta)^{-1}$ or $r \leqslant \bar r_1$ and $h = 1/2$	$\\| x^* - x_n \\| \leqslant 1/\bar h [1 - (1-\delta)\sqrt{1-2h}]^{n+1} \bar\eta (1-\delta)^{-2}$ (5) x^* is unique in Ω_0

Table continued

TABLE III (*continued*)

Theorem	Hypothesis	Conclusion
Th(10): Sensitivity with respect to initial approximation	(1–6) Same as (1–6) in Th(7) (7) $x_0' \in \Omega_0$ (8) $\| x_0' - x_0 \| \leqslant \epsilon \equiv 1 - 2h/4k$ (9) $h \equiv k\eta < 1/2$	(1) MNM and ONM conv. from x_0' (2) $h \geqslant 4\sqrt{2} - 11/2 \approx .16 \Rightarrow \Omega_\epsilon \subset \Omega_{r_0}$ $h < 4\sqrt{2} - 11/2 \approx .16 \Rightarrow \Omega_\epsilon \supset \Omega_{r_0}$
Th(11): Range of x^* in terms of x_0 and x_1	(1–4) Same as (1–4) in Th(7) (5) $\| \Gamma_0 P(x_1) \| \leqslant \eta_1$ $x_1 = x_0 - \Gamma_0 P(x_0)$ (6) $h_1 \equiv K\eta_1(1 - K\eta)^{-2} \leqslant 1/2$	(1–2) Same as (1–2) in Th(7) (3) $\| x^* - x_1 \| \leqslant \dfrac{(1 - \sqrt{1 - 2h})}{h_1} \dfrac{\eta_1}{1 - K\eta}$ (4) Th(11) applies where Th(7) does not if $h_1 \leqslant 1/2$ and $h > 1/2$
Th(12): Mysovskikh extension of Th(7)	(1–2) Same as (1–2) in Th(7) (3) $\| P(x_0) \| \leqslant \eta'$ (4) There exists $\Gamma(x) \equiv [P'(x)]^{-1} \in [Y \to X]$ for every $x \in \Omega_0$ (5) $x \in \Omega_0 \Rightarrow \| \Gamma(x) \| \leqslant B$ (6) $x \in \Omega_0 \Rightarrow \| P''(x) \| \leqslant K'$ (7) $h \equiv B^2 K' \eta' < 2$ (8) $r > r' \equiv B\eta' \sum_{k=0}^{\infty} (h/2)^{2^k - 1}$	(1) There exists $x^* \in \Omega_0$ (2) ONM conv. to x^* for every $x_0 \in \Omega_0$ with $x_n \in \Omega_0$ (3) for ONM $\| x^* - x_n \| \leqslant B\eta' \dfrac{(h/2)^{2^n - 1}}{1 - (h/2)^{2^n}}$
Th(13): Decomposition of $P(x)$	(1) Introduce $\mu \in [Y \to Y]$ such that $P(x)$ becomes $P(x, \mu) = \pi(x) + \mu R(x) = \theta$ where $\pi: X \to Y$, $R: X \to Y$ and $P: X \times Y \to Y$ such that: (2) π and R have cont. second der. in Ω_0 (3) $P(x_0, \theta) = \pi(x_0) = \theta$ (4) There exists $\Gamma_0 \equiv [\pi'(x_0)]^{-1} \in [Y \to X]$ (5) $\| \Gamma_0 \| \leqslant \tilde{B}$ (6) $\| R(x_0) \| \leqslant \tilde{\eta}$, $\| R'(x_0) \| \leqslant \tilde{\alpha}$ (7) $x \in \Omega_0 \Rightarrow \| \pi''(x) \| \leqslant K$, $\| R''(x) \| \leqslant \tilde{L}$ (8) $h_\mu \equiv \dfrac{\tilde{B}^2 \tilde{\eta}(K + \tilde{L} \| \mu \|) \| \mu \|}{(1 - \tilde{\alpha}\tilde{B} \| \mu \|)^2} \leqslant 1/2$ (9) $\tilde{\alpha}\tilde{B} \| \mu \| < 1$	(1) $P(x, \mu) = \theta$ has a sol. $x^*(\mu) \in \Omega_0$ for sufficiently large r.

It should be noted that in Table III, the potential speed of convergence of these methods decreases from the top to the bottom of the table, since one has the possibility of picking a majorant function in Theorem 1 to form sharp bounds, where as in subsequent theorems this choice and other options are sacrificed in favor of standardization of hypothesis by using a widely applicable majorant which in fact may provide only loose bounds.

NEWTON'S METHOD AND $x' = \phi(t, x)$

We shall apply Theorem 7 of Table III to

$$P(x) = x' - \phi(t, x) = 0, \; x(0) = 0$$

in order to obtain a solution $x^* \in C^{(1)}[0, a]$. It is useful to norm this space with

$$\| x \| = \max_{t\in[0,a]} | x(t)| + \lambda \max_{t\in[\theta,a]} | x'(t)|, \qquad \lambda > 0$$

with which, by straightforward manipulations, we obtain

THEOREM. *Let $\phi(u, t)$ be continuous in both its arguments in*

$$\tilde{\Omega} = \{(u, t) | \; t \in [0, a], \; | u - x_0(t)| \leqslant \delta, \; x_0 \in C^{(1)}[0, a], \; x_0(0) = 0\}$$

and have a continuous second derivative in u in this domain: and let

$$(1) \qquad | \phi(x_0(t), t)| \leqslant \eta' \qquad (t \in [0, a])$$

$$(2) \qquad | \phi_u(x_0(t), t)| \leqslant M_1 \qquad (t \in [0, a])$$

$$(3) \qquad | \phi_{uu}(u, t)| \leqslant M_2 \qquad ((u, t) \in \Omega)$$

$$(4) \qquad h_0 = \eta' M_2 a^2 e^{4aM_1} \leqslant 1/2$$

$$(5) \quad \delta > r_0 = (1 - \sqrt{1 - 2h_0}) \, e^{2aM_1} a\eta'/h_0 .$$

Then

$$x'(t) = \phi(x(t), t) \qquad x(0) = 0$$

has a solution $x^*(t)$ in $t \in [0, a]$ where $| x^*(t) - x_0(t)| < r_0$; and a unique solution if

$$h_0 < 1/2 \qquad \text{and} \qquad \delta < r_1 = (1 + \sqrt{1 - 2h_0}) \, e^{2aM_1} a\eta'/h_0 .$$

Furthermore, the modified method and original methods are represented by

$$x'_{n+1} - \phi_u(x_0 , t) \, x_{n+1} = -\phi_u(x_0 , t) \, x_n + \phi(x_n , t),$$

$$x'_{n+1} - \phi_u(x_n , t) \, x_{n+1} = -\phi_u(x_n , t) \, x_n + \phi(x_n , t),$$

with speeds of convergence, respectively, being

$$\| x^* - x_n \| \leqslant (1 - \sqrt{1 - 2h})^{n+1} \eta'/h,$$
$$\| x^* - x_n \| \leqslant 2^{-n}(2h)^{2n} \eta'/h,$$

where

$$h = (ae^{2aM_1} + \lambda\chi)^2 M_2\eta' \leqslant 1/2,$$
$$\chi \geqslant 1 + a \max_{t\in[0,a]} | \phi_u(x_0, t)| \frac{\max | \psi(t)|}{\min | \psi(t)|},$$
$$\psi = \exp \left(\int_0^t \phi_u(x_0(s), s\right) ds.$$

The essential feature of the proof of this theorem is that the inverse of $P'(x_0)$ may be explicitly formed by applying the integrating factor ψ to $P'(x_0) = y$. An outline of the proof may be found in Ref. [3].

Newton's method for $P(x) = x' - \phi(t, x) = 0$, $x(0) = 0$ has the form

$$x'_{n+1} + p_n(t)\, x_{n+1} = q_n(t),$$

where for MNM $p_n(t) = p_0(t)$. Hence we have an explicit expression for $x_{n+1}(t)$,

$$x_{n+1}(t) = \exp\left(-\int p_n(t'')\, dt''\right) \int_0^t q_n(t') \exp\left(\int p_n(t'')\, dt''\right) dt'.$$

Now if $\phi(t, u)$ is approximated by a polynomial or piecewise polynomial, then p_n and q_n are polynomial provided x_n is polynomial. Furthermore, the antiderivative of a polynomial is a polynomial and exponentiation of a polynomial is the composition of two polynomials where $\exp(z)$ has been replaced by a polynomial form. Once again we have chosen $x' = \tanh t \sec x$, $x(0) = 0$, in order to test the ONM, MNM and GNM where $\tanh t$ and $\sec x$ have been replaced by their truncated Maclaurin series of degree N. In these calculations, we set $\Gamma = [P'(z)]^{-1} y$ for $z = t$. For all three algorithms we have taken $x_0(t)$ as parabolas, $x_0(t) = \alpha t + \beta t^2$, in the right half plane and passing through the origin. By virtue of the monotonicity of $| x_0(t)|$, ϕ, ϕ_u, and ϕ_{uu}, we compute

$$| \phi(x_0, t)| \leqslant \tanh a \sec x_0(a) = \eta',$$
$$| \phi_u(x_0, t)| \leqslant \tanh a \sec x_0(a) \tan | x_0(a)| = M_1,$$
$$| \phi_{uu}(u, t)| \leqslant \tanh a \sec x_0(a)(1 + 2 \tan^2 x_0(a)) = M_2,$$
$$h_0(a) = a^2 \tanh^2 a \sec^2 x_0(a)(1 + 2 \tan^2 x_0(a)) \times \exp(4a \tanh a \sec x_0(a) \tan | x_0(a)|),$$

$$\begin{aligned} |\psi| &\leqslant \exp[a \tanh a \sec x_0(a) \tan |x_0(a)|] = e^{aM_1}, \\ |\psi| &\geqslant \exp[a \tanh a \sec x_0(a) \tan |x_0(a)| \geqslant e^{-aM_1}, \\ \chi &= 1 + aM_1 e^{2aM_1}, \\ h(a) &= (ae^{2aM_1} + \lambda\chi(a))^2 M_2(a)\, \eta'(a) \leqslant 1/2, \end{aligned}$$

from which we may write

$$\| x_n - x^* \| \leqslant E(a, n),$$

where $E(a, n)$ denotes the right-hand side of the speed of convergence inequalities. We note that for all $\lambda > 0$ we can find $\bar{a} > 0$ such that $h(\bar{a}) \leqslant 1/2$ (and hence $h_0(\bar{a}) \leqslant 1/2$) and thereby guarantee convergence for values of $a \leqslant \bar{a}$. Computer calculations, for $\lambda \to 0$, indicate convergence of MNM for all $x_0(t) = \alpha t + \beta t^2$ lying within the sector $-1.2t \leqslant x_0(t) \leqslant 1.2t$, $0 \leqslant t$ and within

$$-14t \leqslant x_0(t) \leqslant 14t, 0 \leqslant t$$

for ONM. A numerical analysis of $h(a; \alpha, \beta) \leqslant 1/2$ verify that one should expect such a pattern of convergence. Even for α and β corresponding to $x_0(t)$ far from $x^*(t) = \sin^{-1}(\ln \cosh t)$, convergence in a few iterations to 6 places was observed for $0 \leqslant t \leqslant 1$. We observed earlier that the Maclaurin series diverges for $t > 1.313\cdots$.

Newton's Method and $x'' + x + \phi(t, x, x') = 0$

We shall apply our version of Theorem 8 for $P(x) = \pi(x) + R(x)$ to

$$x'' + x + \phi(t, x, x') = 0, \; x(0) = x'(0) = 0.$$

Solutions to van der Pol's equation, a nonlinear damping problem and a two-point boundary value problem (TPBVP) will be performed and compared to highly accurate numerical integrations, and also analyzed in terms of the theoretical speed of convergence. Note that the conditions $x(0) = x'(0) = 0$ for $\pi(x) = x'' + x$, insures that $[\pi'(x_0)]^{-1}$ is linear, as can be verified by a variation of parameters solution of $\pi'(x_0)\, x = y$. Here we seek solutions in $C^{(2)}([0, a])$ and select the norm

$$\| x \| = \max_{t\in[0,a]} |x(t)| + \max_{t\in[0,a]} |x'(t)| + \lambda \max_{t\in[0,a]} |x''(t)|, \qquad \lambda > 0,$$

and note that

$$[\pi'(x_0)]^{-1} y = \sin t \int_0^t y(s) \cos s\, ds - \cos t \int_0^t y(s) \sin s\, ds$$

from which we can compute, straightforwardly,

$$\|[\pi'(x_0)]^{-1} z\| \leqslant 2a \max |y| + 2a \max |y| + (2a+1) \max |y|$$
$$= k \max_{t\in[0,a]} |y|, \qquad k = 4a + \lambda(2a+1).$$

We next compute the bounds in conditions H(3)′, H(4)′, and H(5)′ where $R(x) = \phi(t, x, x')$ and $\phi = \phi(t, u, v)$. Using the earlier calculations of Frechet derivatives, we have

$$\|[\pi'(x_0)]^{-1} R(x_0)\| \leqslant k \max_{t\in[0,a]} |\phi(t, x_0, x_0')| = \eta$$

$$\|[\pi'(x_0)]^{-1} R(x_0)\, z\| \leqslant k \max_{t\in[0,a]} |\phi_u(t, x_0, x_0')\, z + \phi_v(t, x_0, x_0')\, z'|$$
$$\leqslant k[\max |\phi_u(t, x_0, x_0')|\, |z| + \max |\phi_v(t, x_0, x_0')|\, |z'|]$$
$$\leqslant k[\max |\phi_u(t, x_0, x_0')| + \max |\phi_v(t, x_0, x_0')|]\, \|z\|,$$

where in this last step we have used the definition of $\|z\|$. Hence we may take

$$\|[\pi'(x_0)]^{-1} R(x_0)\| \leqslant k[\max_{t\in[0,a]} |\phi_u(t, x_0, x_0')| + \max_{t\in[0,a]} |\phi_v(t, x_0, x_0')|] = \alpha < 1.$$

Similarly,

$$\|[\pi'(x_0)]^{-1} \pi''(x)\| = |\theta| = 0 = K, \qquad x \in \Omega.$$

$$\|[\pi'(x_0)]^{-1} R''(x)(z, \bar{z})\|$$
$$= k \max_{t\in[0,a]} |\phi_{uu}(t, u, v)\, z\bar{z} + \phi_{uv}(t, u, v)(z\bar{z} + z'\bar{z}) + \phi_{vv}(t, u, v)\, z'\bar{z}|$$
$$\leqslant k[\max |\phi_{uu}(t, u, v)| + 2 \max |\phi_{uv}(t, u, v)| + \max |\phi_{vv}(t, u, v)|]\, \|z\|\, \|\bar{z}\|$$

and hence

$$\|[\pi'(x_0)]^{-1} R''(x)\| \leqslant k[\max_{t\in[0,a]} |\phi_{uu}(t, u, v)|)\, 2 \max_{t\in[0,a]} |\phi_{uv}(t, u, v)|$$
$$+ \max_{t\in[0,a]} |\phi_{vv}(t, u, v)|] = L, \qquad x \in \Omega_0.$$

van der Pol's Equation

We seek a solution to van der Pol's equation $y'' + y + \mu(y^2 - 1)\, y' = 0$, $\mu > 0$ for $y(0) = 1$, $y'(0) = 0$. If we transform y by $y = x + 1$, then we obtain the required boundary conditions $x(0) = x'(0) = 0$ and

$$\phi(t, u, v) = \mu[(u-1)^2 - 1]\, v - 1.$$

Therefore,

$$\eta = k, \alpha = K = 0$$

and

[illegible]

follows from the Levinson–Smith limit cycle Theorem [24]. We next obtain $E(a, n)$, for MNM and for $\lambda \to 0$,

$$h = h_0 = 2\mu k^2(\sigma + 6) = 32a^2\mu(\sigma + 6) \leqslant 1/2.$$

Hence for every μ there exist $a > 0$ such that $h_0 \leqslant 1/2$ and convergence is guaranteed at less for a small interval near the origin. If we had computed $h_0(a)$ directly from Theorem 8 for the particular form of $\phi(t, u, v)$, somewhat larger values of a would satisfy $h_0(a) \leqslant 1/2$. Actually, we will see that the MNM seems to converge rapidly for arbitrarily large a.

If $x_0(t) = 0$, then the MNM for $x = y - 1$ becomes

$$x_{n+1} = \cos t - 1 - \mu \sin t \int_0^t \cos s(x_n + 2)\, x_n x_n'\, ds$$
$$+ \mu \cos t \int_0^t \sin s(x_n + 2)\, x_n x_n'\, ds.$$

The first and second iterations are

$$x_1 = \cos t - 1,$$
$$x_2 = \cos t - 1 - \mu \sin t(\tfrac{1}{4}\cos^4 t - \tfrac{1}{2}\cos^2 t + \tfrac{1}{4})$$
$$+ \mu \cos t(\tfrac{3}{8}t - \tfrac{3}{16}\sin 2t - \tfrac{1}{4}\sin^3 t \cos t).$$

It is apparent, starting with $x_0 = 0$, that all subsequent iterations will involve integrals of the form $t^k \sin^m t \cos^n t$ where sines and cosines of multiple angles are replaced by sums of powers and products of $\sin t$ or $\cos t$. This procedure has been performed by symbol manipulation and then numerically evaluated. Some of these results are summarized in Fig. 5.

The verticle axis shows the norm of the error of x_n, for $\mu = .1, .5, 1, 5, 10, 15$ and of $n = 2, 3, 4, 5$ for $\mu = .1$. Roughly speaking, each successive iteration decreases the error by a factor of 10 and increases the range where $| x_n - x^* | \leqslant 10^{-2}$ by a factor of 2 or 3. Note each iteration contains unbounded terms that accurately account for the nonperiodicity of $x^*(t)$. It is known that for each value of μ there is a periodic solution [24]. Newton's method can be used to obtain this solution by

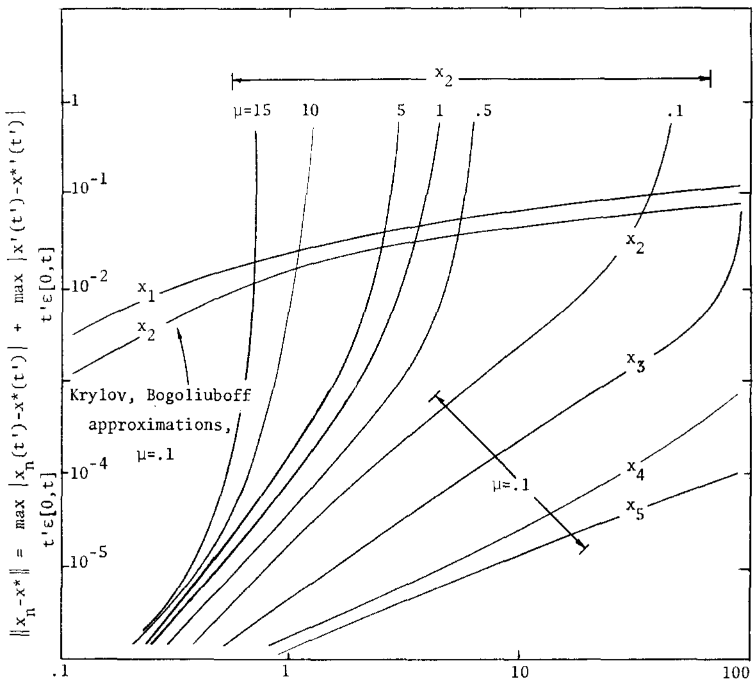

FIG. 5. The solution to van der Pol's equation by Newton's method.

choosing x_0 to be a periodic function with argument ωt and applying conditions $x(\omega t) = x(\omega t + 2\pi)$ and $x'(\omega t) = x'(\omega t + 2\pi)$ in order to determine the solution and the parameter ω symbolically. The resulting manipulations can be identified with those of the classical perturbation method for developing x and ω in power series of μ [25]. For comparison, Fig. 5 shows the error of the first and second Kryloff-Bogoliuboff approximations [26]. Other approximations of van der Pol's equation may be found in [19, 20, 25].

Nonlinear Damping Problem

We next consider the nonlinearly damped system,

$$y'' + \nu^2 y + \mu y'^2 = 0, \nu > 0, \mu > 0 \text{ for } y(0) = 1, y'(0) = 0.$$

We again make the transformation $y = x + 1$ and obtain

$$P(x) = x'' + \nu^2 x + \mu x'^2 + \nu = 0, x(0) = x'(0) = 0.$$

The circumstances of Newton's method are easily shown and the MNM algorithm is

$$x_{n+1} = \cos \nu t - 1 + \frac{\mu}{\nu}\left[\cos \nu t \int_0^t x_n'^2 \sin \nu s\, ds - \sin \nu t \int_0^t x_n'^2 \cos \nu s\, ds\right].$$

Setting $\pi(x) = x'' + \nu^2 x$ and $R(x) = \mu x'^2 + \nu^2$ we obtain, for MNM with $\lambda \to 0$,

$$\| x_n - x^* \| \leqslant \frac{\nu}{8a\mu}(1 - \sqrt{1 - 16\mu a})^{n+1}, \qquad 16\mu a < 1/2$$

from which we are guaranteed convergence for all $\nu > 0$ and μ and a corresponding $a > 0$.

A symbolic evaluation of x_n was performed for $x_0 = lt$. The case $l = 0$ yields sums of terms of the form $t^k \sin^m \nu t^n \cos^n \nu t$ whereas $l \neq 0$ produces sums of $e^{-lt} t^k \sin^m \nu t \cos^n \nu t$, the latter being a result which would be more desirable on physical grounds. The cases $l = 0$ and $l = .1$ for $\nu = 1$ and $\mu = .01$ were examined for $n = 1, 2, 3, 4, 5$ and found to give results similar to those for van der Pol's equation in Fig. 5. Other approximate solutions to this equation are found in [19, 20, 26].

TPBVP

As a last example of Newton's method, we consider $P(x) = x'' - e^x = 0$, $x(0) = x(1) = 0$. Newton's method seems particularly well adapted to TPBVP's since $[P'(x_0)]^{-1}$ is automatically linear, which is not the case ordinarily. For $P(x) = x'' + \phi(t, x, x')$, $\phi = \phi(t, u, v)$ we have

$$P'(x_0)(x) = x'' + \phi_v(t, x_0, x_0')\, x' + \phi_u(t, x_0, x')\, x = y$$

and if z_1 and z_2 are linearly independent homogeneous solutions, then, by variation of parameters and applying $x(0) = x(1) = 0$,

$$[P'(x_0)]^{-1} y = cz_1 \int_0^1 Z_1 y\, ds + c' z_2 \int_0^1 Z_2 y\, ds + z_1 \int_0^t Z_1 y\, ds + z_2 \int_0^t Z_2 y\, ds,$$

where

$$Z_1 = -z_2/(z_1 z_2' - z_1' z_2),\ Z_2 = z_1/(z_1 z_2' - z_1' z_2),$$
$$c = z_1(1)/(z_1(1)\, z_2(0) - z_1(0)), \qquad c' = -c_1 z_1(0)/z_2(0);$$

hence, if $z_1(1)\, z_2(0) - z_1(0) \neq 0$ and $z_2(0) \neq 0$, then $[P'(x_0)]^{-1}$ is linear.

For $P(x) = x'' - e^x$, $x(0) = x(1) = 0$, and choosing $x_0 = 0$, we obtain

$$x_{n+1} = \sinh t \cdot \left(1 - t + \int_1^t e^{x_n(s)}\, ds\right),$$

which converges in one step to the trivial solution $x^*(t) = 0$.

To obtain a nontrivial solution, we note the form of

$$P'(x_0)(x) = x'' - e^{x_0}x,$$

and choose

$$x_0(t) = \ln[kt(t-1) + 1], \qquad k < 4,$$

which yields $[P'(x_0)]^{-1}$ as the solution of

$$x'' - (kt^2 - kt + 1)\, x = y, \qquad x(0) = x(1) = 0.$$

This equation has an ordinary point at $x = 0$ and hence we can obtain two homogeneous solutions of the form $\sum_0^\infty c_i t^i$. The recursion relationship is

$$c_i = \frac{1}{i(i-1)}\, [k(c_{i-4} - c_{i-3}) + c_{i-2}], \qquad i = 2, 3, \ldots,$$

and the homogeneous solutions are

$$z_1 = \sum_0^\infty a_i(k)\, t^i, \qquad z_2 = \sum_0^\infty b_i(k)\, t^i,$$

where, for example,

$$\begin{aligned}
a_0(k) &= 1, & b_0(k) &= 0,\\
a_1(k) &= 0, & b_1(k) &= 1,\\
a_2(k) &= \frac{1}{2}, & b_2(k) &= 0,\\
a_3(k) &= -\frac{1}{3}k, & b_3(k) &= \frac{1}{3},\\
a_4(k) &= \frac{1}{12}\left(k + \frac{1}{2}\right), & b_4(k) &= -\frac{1}{12}k,\\
a_5(k) &= -\frac{1}{24}k, & b_5(k) &= \frac{1}{20}\left(k + \frac{1}{3}\right),\\
a_6(k) &= \frac{1}{90}\left(k^2 + \frac{7}{4}k + \frac{1}{8}\right), & b_6(k) &= \frac{1}{72}k,\\
a_7(k) &= -\frac{k}{504}\left(5k + \frac{1}{3}\right), & b_7(k) &= \frac{1}{126}\left(\frac{1}{4}k^2 + \frac{23}{20}k + \frac{1}{20}\right),\\
&\ \vdots & &\ \vdots
\end{aligned}$$

Z_1 and Z_2 may then be formed by the appropriate multiplication, subtraction, and division of infinite series, and c and c' by summing the rapidly converging sum for $z_1(1)$ and $z_2(1)$. Thus $[P(x_0)]^{-1} y$ is expressed by polynomial operations provided y is a polynomial. Polynomial symbol manipulation with the $[P(x_0)]^{-1}$ just described was applied to the MNM for this problem:

$$x_{n+1} = [P'(x_0)]^{-1} [(kt^2 - kt + 1) x_n - e^{x_n}],$$

where e^{x_n} is replaced by its Maclaurin series in x_n. Values of $k = .1$ and 1 were used. Convergence to 6 places resulted in 2 and 4 iterations, respectively, but only for polynomials of degree larger than 20. Comparison was made with the exact solution,

$$x^*(t) = -\ln 2 + 2 \ln \left\{ c \sec \left[\frac{c}{2} \left(t - \frac{1}{2} \right) \right] \right\}, \qquad c = 1.336055 \cdots.$$

Bellman and Kalaba [23] have used the MNM and the ONM numerically and Varga [27], has used Hermite polynomials to obtain very high accuracy for the solution of this problem.

Symbolic vs. Numerical Solution

The advantages of symbolic solutions as compared to purely numerical solutions are: (1) They are symbolic, (2) they provide qualitative information, and parameter study, (3) they can incorporate automatic formula simplification, and (4) thereby they yield a solution which can be efficiently evaluated numerically and be comparatively devoid of round-off error, (5) symbolic computer programs, say in FORMAC, are simpler and require much less computing art on behalf of the programmer than do strictly numerical programs, (6) the symbolic programs herein required far less machine time than numerical calculations yielding the same accuracy, when compared to Runge–Kutta and relaxation solutions.

Conclusions

On the basis of the above experiments it may be said that formula manipulation of functional algorithms provides a highly efficient method for solving the wide variety of functional equations occurring in engineering and physics. Mundane solution forms, i.e., initial approximations, used here yielded high accuracy in a few iterations. Other solution forms, such as piecewise polynomial or trigonometric forms [8], would surely result in substantially sharper results. However, the purpose of this paper was to examine the solution method and to show that, even using mundane solution forms, accurate solutions are readily obtainable. It is expected that solutions, by these methods, of partial differential equations, integral equations, integrodifferential equations and differential-difference equations would result in corresponding accuracy. An interesting prospect would be to linearize a nonlinear partial differential equation by Newton's method and to attempt to solve the successive linear partial differential equations by SD or SA. In addition to the SA, SD, MNM, ONM and GNM, other functional algorithms may be used; for example, the generalized regula-falsi method [6], higher order Newton's method [28] and the results of Antosiewicz [18] and others [2, 26, 27, 29]. Lastly, we should recognize the heuristic value of these methods in that they provide solutions to a far wider class of problems than rigorous theory would indicate.

References

1. D. G. Bobrow (Ed.), "Symbol Manipulation Languages and Techniques," Proceedings of the Working Committee on Symbol Manipulation Languages, North-Holland, Amsterdam, 1969.
2. S. R. Petrick (Ed.), "Proceedings of the Second Symposium on Symbolic and Algebraic Manipulation," March 23–25, 1971, Association for Computing Machinery, Los Angeles, 1971.
3. L. V. Kantorovich and G. P. Akilov, "Functional Analysis in Normed Spaces," Pergamon Press, New York, (1964).
4. H. A. Antosiewicz and W. C. Rheinbolt, Numerical analysis and functional analysis, *in* "Survey of Numerical Analysis" (J. Todd, Ed.), McGraw-Hill, New York 1962.
5. A. Deprit, J. Henrard, and A. Rom, Lunar ephemeris: Delaunay's theory revisited, *Science* **168** (1970), 1569–1570.
6. L. Collatz, "Functional Analysis and Numerical Mathematics," Academic Press, New York, 1966.
7. R. Caccioppoli, Sugli elementi uniti delle transformazione funzionali: Osservazione sui problemi di valori ai limiti, *Acc. Maz. Lincei*, **6** (1931), 498–502.
8. T. N. E. Greville (Ed.), "Theory and Applications for Spline Functions," Academic Press, New York, 1969.
9. L. E. Elsgolc, "Calculus of Variations," Pergamon Press, London, 1962.
10. S. G. Mikhlin, "The Problem of the Minimum of a Quadratic Functional," Holden-Day, San Francisco, 1965.
11. M. M. Vainberg, "Variational Methods for the Study of Nonlinear Operators," Holden-Day, San Francisco, 1964.
12. L. V. Kantorovich and V. I. Krylov, "Approximate Methods of Higher Analysis," P. Noordhoff, Groningen, Netherlands, 1964.
13. L. A. Lusternik and V. J. Sobolev, "Elements of Functional Analysis," Ungar, New York, 1961.
14. J. N. Hanson and A. D. Waren, Experiments in solving large and ill-conditioned algebraic linear systems by Kantorovich's p-step steepest descentmet hod, to appear in the Cleveland State University Mathematics Research Report, 1971–72.

15. L. V. Kantorovich, "Functional Analysis and Applied Mathematics," translation, National Bureau of Standards, Washington, D.C., 1953.
16. A. M. Bruckner and J. L. Leonard, Derivatives, *Amer. Math. Mo.* **73** (1966), 24–56.
17. *ibid*, pp. 63–76.
18. H. A. Antosiewicz, Newton's method and boundary value problems, *J. Comput. System Sci.* **2** (1968), 177–202.
19. R. E. Bellman, "Methods of Nonlinear Analysis," Academic Press, New York, 1970.
20. J. P. LaSalle and S. Lefschetz, "Nonlinear Differential Equations and Nonlinear Mechanics," Academic Press, New York, 1963.
21. W. F. Ames, "Nonlinear Ordinary Equations in Transport Processes," Academic Press, New York, 1968.
22 E. S. Lee, "Quasilinearization and Invariant Imbedding," Academic Press, New York, 1968.
23. R. E. Bellman and R. E. Kalaba, "Quasilinearization and Nonlinear Boundary Value Problems," American Elsevier, New York, 1965.
24. G. Birkhoff and G. Rota, "Ordinary Differential Equations," Blaisdell, Waltham, Mass., 1959.
25. R. E. Bellman, "Perturbation Techniques in Mathematics, Physics and Engineering," Holt, Rinehart and Winston, New York, 1966.
26. N. Kryloff and N. Bogoliuboff, "Introduction to Nonlinear Mechanics," Annals of Mathematics Studies, No. 11, Princeton Univ. Press, Princeton, N.J., 1947.
27. R. S. Varga, Accurate numerical methods for nonlinear boundary value problems, *in* "Numerical Solution of Field Problems in Continuum Physics," American Mathematical Society, Providence, R.I., 1970.
28. W. E. Bosarge and P. L. Falb, "Infinite Dimensional Multipoint Methods and the Solution of Two Point Boundary Value Problems," Center for Dynamical Systems, Brown Univ., Providence, R.I., 1968.
29. S. K. Ferriera, L. H. Sobel, and B. G. Wrenn, "On the Application of FORMAC Computer Programs to Aid in Symbolic Formulation and Solution of Engineering and Scientific Problems," Lockheed Co. Research paper, Palo Alto, California, June 1967.